

TOPOLOGY OF A CLASS OF $p2$ -CRYSTALLOGRAPHIC REPLICATION TILES

BENOÎT LORIDANT AND SHU-QIN ZHANG

ABSTRACT. We study the topological properties of a class of planar crystallographic replication tiles. Let $M \in \mathbb{Z}^{2 \times 2}$ be an expanding matrix with characteristic polynomial $x^2 + Ax + B$ ($A, B \in \mathbb{Z}$, $B \geq 2$) and $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, M\mathbf{v})$ are linearly independent. Then the equation

$$MT + \frac{B-1}{2}\mathbf{v} = T \cup (T + \mathbf{v}) \cup (T + 2\mathbf{v}) \cup \cdots \cup (T + (B-2)\mathbf{v}) \cup (-T)$$

defines a unique nonempty compact set T satisfying $\overline{T^o} = T$. Moreover, T tiles the plane by the crystallographic group $p2$ generated by the π -rotation and the translations by integer vectors. It was proved by Leung and Lau in the context of self-affine lattice tiles with collinear digit set that $T \cup (-T)$ is homeomorphic to a closed disk if and only if $2|A| < B + 3$. However, this characterization does not hold anymore for T itself. In this paper, we completely characterize the tiles T of this class that are homeomorphic to a closed disk.

1. INTRODUCTION

A *crystallographic replication tile* with respect to a crystallographic group $\Gamma \subset \text{Isom}(\mathbb{R}^n)$ is a nonempty compact set $T \subset \mathbb{R}^n$ that is the closure of its interior ($\overline{T^o} = T$) and satisfies the following properties.

- (i) There is an expanding affine mapping $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $g \circ \Gamma \circ g^{-1} \subset \Gamma$, and a finite collection $\mathcal{D} \subset \Gamma$ called *digit set* such that

$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T).$$

- (ii) The family $\{\gamma(T); \gamma \in \Gamma\}$ is a *tiling* of \mathbb{R}^n . In other words, $\mathbb{R}^n = \bigcup_{\gamma \in \Gamma} \gamma(T)$ and $\gamma(T^o) \cap \gamma'(T^o) = \emptyset$ for distinct elements $\gamma, \gamma' \in \Gamma$.

There is a vast literature dealing with the lattice case, *i.e.*, when Γ is isomorphic to \mathbb{Z}^n : criteria exist to check basic properties, such as the tiling property [11], connectedness [9] or, in the planar case ($n = 2$), homeomorphy to a closed disk (*disk-likeness*). For instance, Bandt and Wang recognize disk-like self-affine lattice tiles by the number and location of the neighbors in the tiling [4], and Lau and Leung characterize all the disk-like tiles among the class of self-affine lattice tiles with collinear digit set [12]. A powerful tool in the study of topological properties is the *neighbor graph*: it gives a precise description of the boundary of the tile in terms of a *graph directed iterated function system (GIFS)*. Akiyama and the first author elaborated a boundary parametrization method by making extensive use of the neighbor graph [2]. Algorithms allow to determine the neighbor graph for any given tile T [16], while it is usually difficult to deal with infinite classes of tiles. However, Akiyama and Thuswaldner computed the neighbor graph for an infinite

Date: November 16, 2016.

2010 Mathematics Subject Classification. Primary: 28A80. Secondary: 52C20, 20H15.

Key words and phrases. Self-affine sets, Tilings, Crystallographic $p2$ -tiles, Neighbor sets.

Supported by project I1136 and by the doctoral program W1230 granted by the Austrian Science Fund (FWF).

class of planar self-affine lattice tiles associated with canonical number systems and used it to characterize the disk-like tiles among this class [3]. Methods relying on the neighbor graph were extended to crystallographic replication tiles in [14, 15].

If T is a crystallographic replication tile, the associated digit set \mathcal{D} must be a complete set of right coset representatives of the subgroup $g \circ \Gamma \circ g^{-1}$. On the other side, if T is a nonempty compact set $T \subset \mathbb{R}^n$ satisfying (i) and \mathcal{D} is a complete set of right coset representatives of the subgroup $g \circ \Gamma \circ g^{-1}$, Gelbrich proves that there is a subset $\Gamma' \subset \Gamma$ called *tiling set* such that the family $\{\gamma(T); \gamma \in \Gamma'\}$ is a tiling of \mathbb{R}^n . Under these conditions, it is not known in general whether the tiling set Γ' is a *subgroup* of the crystallographic group Γ , contrary to the lattice case (see [10]). However, the first author defined in [13] the *crystallographic number systems*, in analogy to the canonical number systems from the lattice case (see e.g. [8]). This gives a way to produce classes of crystallographic replication tiles whose tiling set is the whole group Γ . An infinite class of examples given in [13] reads as follows. Let p_2 be the planar crystallographic group generated by the translations $a(x, y) = (x + 1, y)$, $b(x, y) = (x, y + 1)$ and the π -rotation $c(x, y) = (-x, -y)$. Moreover, for $A, B \in \mathbb{Z}$ satisfying $|A| \leq B \geq 2$, let g be the expanding mapping defined on \mathbb{R}^2 by

$$g(x, y) = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}.$$

Then the equation

$$g(T) = T \cup \left(T + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cup \cdots \cup \left(T + \begin{pmatrix} B-2 \\ 0 \end{pmatrix} \right) \cup (-T)$$

defines a crystallographic replication tile whose tiling set is the whole group p_2 . This tiling property follows from the crystallographic number system property only for $A \geq -1$, as stated in [13], but we will deduce it for all values of A . Moreover, we will obtain topological information on T by comparing it with the self-affine lattice tile T^l defined by

$$\begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} T^\ell = T^\ell \cup \left(T^\ell + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cup \cdots \cup \left(T^\ell + \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right).$$

In fact, for fixed A and B , the tile T^l is a translation of $T \cup (-T)$, as shown in [13]. It follows from Leung and Lau's result [12] on self-affine tiles with collinear digit set that T^l is disk-like if and only if $2|A| - B < 3$. However, it was noticed in [13] that it can happen that T^l is disk-like while T is not disk-like (see the examples of Figure 9 and Figure 10). The current paper will establish exactly for which parameters A, B this phenomenon occurs. For $2|A| - B < 3$, the associated lattice tile T^l is disk-like and a result of Akiyama and Thuswaldner [3] on canonical number system tiles will allow us to estimate the set of neighbors of T . Finding out the disk-like tiles for parameters satisfying $2|A| - B < 3$ will then rely on the construction of the associated neighbor graphs for the whole class. For $2|A| - B \geq 3$, a purely topological argument will enable us to prove that the associated tiles are not disk-like.

Our results easily generalize to a broader class of crystallographic replication tiles, closely related to the class of self-affine tiles with collinear digit set as studied by Leung and Lau in [12]. Therefore, we are able to show the following classification theorem.

Theorem. *Let $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $B \geq 2$, $M \in \mathbb{Z}^{2 \times 2}$ a matrix with characteristic polynomial $x^2 + Ax + B$ and let $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, M\mathbf{v})$ are linearly independent. Let T be the crystallographic replication tile defined by*

$$MT + \frac{B-1}{2}\mathbf{v} = T \cup (T + \mathbf{v}) \cup (T + 2\mathbf{v}) \cup \cdots \cup (T + (B-2)\mathbf{v}) \cup (-T).$$

Then the following statements hold.

- Suppose that $2|A| - B \geq 3$. Then T is not disk-like.
- Suppose that $2|A| - B < 3$. Then one of the following cases occurs:
 - (1) If $A \in \{-2, -1, 0, 1\}$, $B \geq 2$ or $A = 2$, $B = 2$, then T is disk-like.
 - (2) If $A \geq 2$, $B \geq 3$ or $A \leq -3$, $B \geq 4$, then T is not disk-like.

The paper is organized as follows. In Section 2, we give basic definitions on crystallographic groups and general properties of the class of crystallographic replication tiles under consideration. Sections 3 and 4 are devoted to the construction of the neighbor graphs for part of this class. They will be the main tool for our topological study. In Section 5 and Section 6, we characterize the disk-like tiles among our class for the range of parameters A, B satisfying $2|A| - B < 3$. In Section 7, we show that T is not disk-like for all parameters satisfying $2|A| - B \geq 3$. Finally, Section 8 illustrates the theorem by examples.

2. PRELIMINARIES

2.1. Basic definitions. Let us recall some definitions and facts about tilings and crystallographic replication tiles (*crystiles* for short).

A tiling of \mathbb{R}^2 is a cover of the space by nonoverlapping sets, *i.e.*, such that the interiors of two distinct sets of the cover are disjoint. Some particular tilings use a single tile T with $\overline{T^\circ} = T$ and a family Γ of isometries of \mathbb{R}^2 such that

$$\mathbb{R}^2 = \bigcup_{\gamma \in \Gamma} \gamma(T).$$

Assume that Γ contains id , the identity map of \mathbb{R}^2 . Then $T = id(T)$ is called the *central tile* of the tiling. Also, two distinct tiles are said to be *neighbors* if they have common points. The neighbor set of T is then given by

$$\mathcal{S} = \{\gamma \in \Gamma \setminus \{id\}; \gamma(T) \cap T \neq \emptyset\}.$$

It is symmetric and it generates Γ ($\Gamma = \langle \mathcal{S} \rangle$). The tiles considered in this paper will be compact and the tilings locally finite, *i.e.*, every compact set intersects finitely many tiles of the tiling. Thus \mathcal{S} is always a finite set here.

Among the possible neighbors of a tile, we consider the following two kinds of neighbors. Two neighbors are called *vertex neighbors* if they have only one common point. Two neighbors are *adjacent neighbors* if the interior of their union contains a point of their intersection. The *adjacent neighbor set* $\mathcal{A} \subset \mathcal{S}$ is then defined as the set of adjacent neighbors of the identity:

$$\mathcal{A} = \{\gamma \in \mathcal{S}; T \cap \gamma(T) \cap (T \cup \gamma(T))^\circ \neq \emptyset\}.$$

The neighbor (resp. adjacent neighbor) set of a tile $\gamma(T)$ ($\gamma \in \Gamma$) is equal to $\gamma\mathcal{S}$ (resp. $\gamma\mathcal{A}$).

We will deal with families Γ of isometries that are *crystallographic groups* in dimension 2, *i.e.*, discrete cocompact subgroups Γ of the group $\text{Isom}(\mathbb{R}^2)$ of all isometries on \mathbb{R}^2 with respect to some metric. By a theorem of Bieberbach (see [5]), a crystallographic group Γ in dimension 2 contains a group Λ of translations isomorphic to the lattice \mathbb{Z}^2 , and the quotient group Γ/Λ , called *point group*, is finite. There are 17 nonisomorphic such groups. However, in this paper, we will mainly consider the following crystallographic $p2$ -groups.

Definition 2.1. Let $a(x, y) = (x + 1, y)$, $b(x, y) = (x, y + 1)$, $c(x, y) = (-x, -y)$. Then a $p2$ -group is a group of isometries of \mathbb{R}^2 isomorphic to the subgroup of $\text{Isom}(\mathbb{R}^2)$ generated by the translations a, b and the π -rotation c .

For example, the standard $p2$ -group Γ has the form

$$(2.1) \quad \Gamma = \{a^p b^q c^r; p, q \in \mathbb{Z}, r \in \{0, 1\}\},$$

and it is a crystallographic group. We will call a tiling with respect to a $p2$ -group a $p2$ -tiling, and a tiling with respect to a lattice group (*i.e.*, for which the point group only contains the class of the identity map of \mathbb{R}^2) a *lattice tiling*.

We will be concerned with self-replicating tiles constructed in the following way. We refer the reader to [6, 15] for further information about these tiles.

Definition 2.2. A *crystallographic replication tile* with respect to a crystallographic group Γ is a compact nonempty set $T \subset \mathbb{R}^n$ with the following properties:

- The family $\{\gamma(T); \gamma \in \Gamma\}$ is a tiling of \mathbb{R}^n .
- There is an expanding affine map $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $g \circ \Gamma \circ g^{-1} \subset \Gamma$ and there exists a finite collection $\mathcal{D} \subset \Gamma$ called *digit set* such that

$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T).$$

2.2. Lattice tiling and $p2$ -tiling. From now on, Γ is the standard $p2$ -group defined (2.1). We recall that an expanding affine map g in \mathbb{R}^n has the form $g(x) = Mx + t$, where $t \in \mathbb{R}^n$ and M is an $n \times n$ -matrix whose eigenvalues all have modulus greater than 1.

We consider a special class of $p2$ -crystallographic replication tiles, closely related to the class of self-affine tiles with collinear digit set studied by Leung and Lau in [12]. For $A, B \in \mathbb{Z}$, $B \geq 2$, let $\tilde{M} \in \mathbb{Z}^{2 \times 2}$ be a matrix with characteristic polynomial $x^2 + Ax + B$. Then \tilde{M} is expanding, *i.e.*, its eigenvalues are greater than 1 in modulus, if and only if $|A| \leq B$. Moreover, let $\mathbf{v} \in \mathbb{Z}^2$ such that $(\mathbf{v}, \tilde{M}\mathbf{v})$ are linearly independent and $\tilde{a}(x, y) = (x, y) + \mathbf{v}$. We set $\tilde{g}(\mathbf{x}) = \tilde{M}\mathbf{x} + \frac{B-1}{2}\mathbf{v}$. Then one can check that the digit set

$$\tilde{\mathcal{D}} = \{id, \tilde{a}, \dots, \tilde{a}^{B-2}, c\}$$

is a complete set of right coset representatives of the subgroup $\tilde{g} \circ \Gamma \circ \tilde{g}^{-1}$. Therefore, by a result of Gelbrich [6], the equation

$$\tilde{g}(\tilde{T}) = \bigcup_{\delta \in \tilde{\mathcal{D}}} \delta(\tilde{T})$$

defines a unique nonempty compact set $\tilde{T}(A, B) = \tilde{T}$ satisfying $\overline{\tilde{T} \circ} = \tilde{T}$. The purpose of this paper is the topological study of the tiles \tilde{T} . In fact, we can reduce this study to the following subclass. Let

$$(2.2) \quad g(x, y) = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}$$

and

$$(2.3) \quad \begin{aligned} \mathcal{D} &= \{id, a, \dots, a^{B-2}, c\} & \text{if } B \geq 3, \\ \mathcal{D} &= \{id, c\} & \text{if } B = 2, \end{aligned}$$

which is a complete right residue system of $\Gamma/g\Gamma g^{-1}$. We denote by $T(A, B) = T$ the associated tile satisfying

$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T).$$

Lemma 2.3. *In the above notations, let C denote the matrix of change of base from the canonical base to the base $(\mathbf{v}, \tilde{M}\mathbf{v})$. Then*

$$\tilde{T} = CT.$$

Proof. We have by definition

$$\tilde{M}\tilde{T} + \frac{B-1}{2}\mathbf{v} = \tilde{T} \cup (\tilde{T} + \mathbf{v}) \cup \dots \cup (\tilde{T} + (B-2)\mathbf{v}) \cup (-\tilde{T}).$$

Using the equality $\tilde{M} = CM C^{-1}$ and multiplying the above equation by C^{-1} , we obtain

$$MC^{-1}\tilde{T} + \left(\frac{B-1}{2} \right) = C^{-1}\tilde{T} \cup \left(C^{-1}\tilde{T} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cup \dots \cup \left(C^{-1}\tilde{T} + \begin{pmatrix} B-2 \\ 0 \end{pmatrix} \right) \cup (-C^{-1}\tilde{T}).$$

By uniqueness of the IFS-attractor, this leads to $C^{-1}\tilde{T} = T$. \square

By this lemma, the topology of \tilde{T} is the same as the topology of T , that is why the proofs will be written for the class of tiles defined by (2.2) and (2.3). The relation to self-affine tiles with collinear digit set now reads as follows. Let $M = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix} \in \mathbb{Z}^{2 \times 2}$ and

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\}.$$

We denote by $T^\ell(A, B) = T^\ell$ the associated lattice tile satisfying

$$MT^\ell = \bigcup_{d \in \mathcal{N}} (T^\ell + d).$$

Note that the crystallographic data $(p2, g, \mathcal{D})$ is very similar to the lattice data $(\mathbb{Z}^2, M, \mathcal{N})$. However, in the lattice case, we often prefer to consider the above translation vectors rather than the translation mappings id, a, \dots, a^{B-1} . Moreover, [13] also gave the following correspondence between the crystallographic tiles and the associated lattice tiles of the above class.

Lemma 2.4. *With the above data, let T satisfy $g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T)$ and T^ℓ satisfy $MT^\ell = \bigcup_{d \in \mathcal{N}} (T^\ell + d)$. Then*

$$(2.4) \quad T^\ell = T \cup (-T) + (M - I_2)^{-1} \begin{pmatrix} B-1 \\ 0 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix.

Hereafter, we denote the crystallographic tile and lattice tile associated with the above data $(p2, g, \mathcal{D})$ and $(\mathbb{Z}^2, M, \mathcal{N})$ by T and T^ℓ , respectively.

Lemma 2.5. *T is a crystallographic replication tile.*

Proof. We only need to prove that the family $\{\gamma(T); \gamma \in \Gamma\}$ is a tiling of \mathbb{R}^2 . We recall that T has nonempty interior by a result of Gelbrich [6], because \mathcal{D} is a complete set of right coset representatives of $g \circ \Gamma \circ g^{-1}$. Now, for $A \geq -1$, the family $\{T^\ell + z; z \in \mathbb{Z}^2\}$ is a tiling of \mathbb{R}^2 , since the tile T^ℓ is associated to a quadratic canonical number system (see e.g. [3]). This also holds for the tiles T^ℓ with $A \leq 0$, as it is mentioned in [1] that changing A to $-A$, for a fixed B , results in an isometric transformation for the associated tiles T^ℓ (see Equation (6.1)). Therefore, by Lemma 2.4, we just need to show that T and $c(T) = -T$ do not overlap. This follows from the fact that T has nonempty interior and satisfies the set equation

$$T = g^{-1}(T) \cup g^{-1}(-T) \cup g^{-1} \circ a(T) \dots \cup g^{-1} \circ a^{B-2}(T).$$

Indeed, each of the B sets on the right side of this equation has two-dimensional Lebesgue measure α/B , where $\alpha > 0$ is the two-dimensional Lebesgue measure of T . The total measure of the right side being equal to α , the sets can not overlap. \square

Note that for $-1 \leq A \leq B$, the above lemma is also a consequence of the crystallographic number system property [13].

Remark 2.6. In the above proof, we mentioned the easy relation (6.1) between the lattice tiles T^ℓ associated to A and $-A$. It turns out that no such easy relation can be found for the corresponding tiles T , and the topology may become different when changing A to $-A$ (see Section 6, Figure 6).

For the lattice data $(\mathbb{Z}^2, M, \mathcal{N})$, the following proposition is proved by Leung and Lau [12].

Proposition 2.7. *Let A and B satisfy $|A| \leq B$ and $B \geq 2$. Then T^ℓ is homeomorphic to a closed disk if and only if $2|A| < B + 3$.*

3. THE NEIGHBOR SET OF T FOR $A \geq -1$ AND $2A < B + 3$

For the sake of simplicity, in Sections 3, 4 and 5 we will now restrict to the case $A \geq -1$ and $2A < B + 3$ and indicate in Section 6 the method to get the results for $A \leq -2$.

In this section, we will derive an “approximation” of the neighbor set \mathcal{S} for $A \geq -1$, $2A < B + 3$ from the relationship between the neighbor set of T and the neighbor set of T^ℓ . Akiyama and Thuswaldner prove the following characterization of the neighbors of T^ℓ in [3].

Proposition 3.1. *Let \mathcal{S}^ℓ denote the neighbor set of T^ℓ . If $2A < B + 3$ and $A \neq 0$, then $\sharp\mathcal{S}^\ell = 6$. In particular, if $A > 0$, then*

$$\mathcal{S}^\ell = \{a^A b, a^{A-1} b, a, a^{-1}, a^{-A} b^{-1}, a^{-A+1} b^{-1}\};$$

if $A = -1$, we have

$$\mathcal{S}^\ell = \{a^{-1} b, b, a, a^{-1}, ab^{-1}, b^{-1}\};$$

if $A = 0$, we have

$$\mathcal{S}^\ell = \{a, a^{-1}, ab, a^{-1} b, ab^{-1}, a^{-1} b^{-1}, b, b^{-1}\}.$$

The following lemma gives a first coarse estimate of the neighbor set of T in terms of the neighbor set of T^ℓ .

Lemma 3.2. *Let $\mathcal{S}, \mathcal{S}^\ell$ be the neighbor sets of T and T^ℓ , respectively, then \mathcal{S} is a subset of $\mathcal{S}^\ell \cup \{c\} \cup \mathcal{S}^\ell c$, where $\mathcal{S}^\ell c = \{s \circ c; s \in \mathcal{S}^\ell\}$.*

Proof. Using Lemma 2.4, we know that the lattice tile is a translation of the union $T \cup c(T)$. Then it is easy to see that all possible elements of the neighbor set of T are included in the union of the neighbor set of T^ℓ , the π -rotation of the neighbor set of T^ℓ and the π -rotation itself. \square

From the above lemma, we know an upper bound for the number of neighbors of the $p2$ -tile T . The following proposition from [14] gives a lower bound for this number.

Proposition 3.3. *In a lattice tiling or a $p2$ -tiling of the plane, each tile has at least six neighbors.*

Let us now give the definition of the neighbor graph.

Definition 3.4. ([15]) Let Γ be a group of isometries on \mathbb{R}^2 , let g be an expanding map and let \mathcal{D} be a digit set associated with a given tile. For $\Omega \subset \Gamma$ we define the graph $G(\Omega)$ as follows. The states of $G(\Omega)$ are the elements of Ω , and there is an edge

$$\gamma \xrightarrow{\delta|\delta'} \gamma' \quad \text{iff } \delta^{-1} g \gamma g^{-1} \delta' = \gamma' \text{ with } \gamma, \gamma' \in \Omega \text{ and } \delta, \delta' \in \mathcal{D}.$$

The *neighbor graph* $G(\mathcal{S})$ is very important in the present paper.

Recall that the neighbor set of T is defined by $\mathcal{S} = \{\gamma \in \Gamma \setminus \{id\}; T \cap \gamma(T) \neq \emptyset\}$. Set $B_\gamma = T \cap \gamma(T)$ for $\gamma \in \Gamma$. The nonoverlapping property yields for the boundary of T that $\partial T = \bigcup_{\gamma \in \mathcal{S}} B_\gamma$. Moreover using the above notation, the sets B_γ satisfy the set equation ([15])

$$B_\gamma = \bigcup_{\substack{\delta \in \mathcal{D}, \gamma' \in \mathcal{S}, \\ \exists \delta' \in \mathcal{D}, \gamma \xrightarrow{\delta|\delta'} \gamma' \in G(\mathcal{S})}} g^{-1}\delta(B_{\gamma'}).$$

The following characterization is from [15].

Characterization 3.5. Let t be a point in \mathbb{R}^n , $(\delta_j)_{j \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ and $\gamma \in \mathcal{S}$. Then the following assertions are equivalent.

- $x = \lim_{n \rightarrow \infty} g^{-1}\delta_1 \dots g^{-1}\delta_n(t) \in B_\gamma$.
- There is an infinite walk in $G(\mathcal{S})$ of the shape

$$\gamma \xrightarrow{\delta_1|\delta'_1} \gamma_1 \xrightarrow{\delta_2|\delta'_2} \gamma_2 \xrightarrow{\delta_3|\delta'_3} \dots$$

for some $\gamma_i \in \mathcal{S}$ and $\delta'_i \in \mathcal{D}$.

This means that for each $\gamma \in \mathcal{S}$, there is at least one infinite walk in $G(\mathcal{S})$ starting from the state γ . We use this information to refine the estimate of the neighbor set of T (compare with Lemma 3.2).

Lemma 3.6. Let \mathcal{S} be the neighbor set of the tile T with respect to $(p2, g, \mathcal{D})$. Let $\mathcal{S}' = \mathcal{S}^\ell \cup \{c\} \cup \mathcal{S}^\ell c$. Then the following statements hold.

- (1) For $A > 0$, $\mathcal{S} \subset \mathcal{S}' \setminus \{a^A b, a^{-A} b^{-1}, a^{-A} b^{-1} c\}$;
- (2) For $A = -1$, $\mathcal{S} \subset \mathcal{S}' \setminus \{a^{-1} b, b, ab^{-1}, b^{-1}, ab^{-1} c, b^{-1} c\}$;
- (3) For $A = 0$, $\mathcal{S} \subset \mathcal{S}' \setminus \{ab, a^{-1} b^{-1}, ab^{-1}, a^{-1} b, b, b^{-1}, a^{-1} b^{-1} c, ab^{-1} c, b^{-1} c\}$.

In particular, \mathcal{S} has at least 6 but not more than 10 elements.

Proof. We know that $G(\mathcal{S})$ is a subgraph of $G(\mathcal{S}')$ by Lemma 3.2. The definition of the edges requires to calculate $g\mathcal{S}'g^{-1} = \{g\gamma g^{-1}; \gamma \in \mathcal{S}'\}$ at first. Let p and q be arbitrary elements in \mathbb{Z} . Recall that g has the form (2.2). Then

$$(3.1) \quad ga^p b^q g^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -qB \\ p - qA \end{pmatrix},$$

$$(3.2) \quad ga^p b^q c g^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1-q)B - 1 \\ p - qA \end{pmatrix}.$$

Thus the following relations hold:

$$ga^A b g^{-1} = a^{-B}, \quad ga^{-A} b^{-1} g^{-1} = a^B, \quad ga^{-A} b^{-1} c g^{-1} = a^{2B-1} c.$$

We claim that there are no edges starting from the states $a^A b$, $a^{-A} b^{-1}$, and $a^{-A} b^{-1} c$.

Indeed, for $\delta, \delta' \in \mathcal{D}$,

$$\delta^{-1} ga^A b g^{-1} \delta' = \delta^{-1} a^{-B} \delta' = \begin{cases} a^{-B}, & \delta = \delta' = id; \\ a^B, & \delta = \delta' = c; \\ a^{-B} c, & \delta = id, \delta' = c; \\ a^B c, & \delta = c, \delta' = id; \\ a^{B-p+q}, & \delta = a^p, \delta' = a^q, 1 \leq p, q \leq B-2. \end{cases}$$

Therefore, $\delta^{-1} ga^A b g^{-1} \delta'$ is not an element of \mathcal{S}' , which means that there is no edge starting from $a^A b$. The computation is similar for $a^{-A} b^{-1}$, $a^{-A} b^{-1} c$. Hence, we obtain that $a^A b$, $a^{-A} b^{-1}$, $a^{-A} b^{-1} c$ are not elements of \mathcal{S} by Characterization 3.5, which proves Item (1).

For $A = -1$, by (3.1) and (3.2) we know that

$$\begin{aligned} ga^{-1}bg^{-1} &= a^{-B}, & gbg^{-1} &= a^{-B}b, & gab^{-1}g^{-1} &= a^B, \\ gb^{-1}g^{-1} &= a^Bb^{-1}, & gab^{-1}cg^{-1} &= a^{2B-1}c, & gb^{-1}cg^{-1} &= a^{2B-1}b^{-1}c. \end{aligned}$$

Similar computations as above show that there is no edge starting from the states removed from \mathcal{S}' in Item (2).

For $A = 0$, we can also show that there is no edge starting from the states removed from \mathcal{S}' in Item (3), since

$$\begin{aligned} ga^{-1}bg^{-1} &= a^{-B}b^{-1}, & gbg^{-1} &= a^{-B}, & gab^{-1}g^{-1} &= a^Bb, \\ gb^{-1}g^{-1} &= a^B, & gab^{-1}cg^{-1} &= a^{2B-1}bc, & gb^{-1}cg^{-1} &= a^{2B-1}c, \\ gabg^{-1} &= a^{-B}b, & ga^{-1}b^{-1}g^{-1} &= a^Bb^{-1}, & ga^{-1}b^{-1}cg^{-1} &= a^{2B-1}b^{-1}c. \end{aligned}$$

Finally, by Proposition 3.3 and the above discussion, we obtain that the neighbor set of the crystile has at least 6 but not more than 10 elements because $\#\mathcal{S}' = 13$ by Lemma 3.2. \square

4. THE NEIGHBOR GRAPH OF T FOR $A \geq -1$ AND $2A < B + 3$

In this section, we explicitly construct the neighbor graph. Throughout the whole section, we restrict to the case $A \geq -1$ and $2A < B + 3$. In Lemma 3.6, we denoted by \mathcal{S}' the set $\mathcal{S}' = \mathcal{S}^\ell \cup \{c\} \cup \mathcal{S}^\ell c$. Now for $A > 0$, let $\mathcal{S}'' = \mathcal{S}' \setminus \{a^A b, a^{-A} b^{-1}, a^{-A} b^{-1} c\}$, that is,

$$(4.1) \quad \mathcal{S}'' = \{a^{A-1}b, a, a^{-1}, b^{-1}, a^{1-A}b^{-1}, c, a^A bc, a^{A-1}bc, ac, a^{-1}c, a^{1-A}b^{-1}c\}.$$

For $A = 0$, we set

$$(4.2) \quad \mathcal{S}'' = \{a, a^{-1}, c, ac, a^{-1}c, a^{-1}bc, bc, abc\},$$

and for $A = -1$,

$$(4.3) \quad \mathcal{S}'' = \{a, a^{-1}, c, ac, a^{-1}c, a^{-1}bc, bc\}.$$

By Lemma 3.2, we know that $\mathcal{S} \subset \mathcal{S}''$. We call the graph $G(\mathcal{S}'')$ the *pseudo-neighbor graph*. Tables 1, 2 and 3 show all information on $G(\mathcal{S}'')$. The last column indicates the parameters A, B for which these edges exist. Furthermore, the pseudo-neighbor graphs for the cases $A \geq 3, B \geq 5$ are depicted in Figure 1. The edges named by (1), \dots , (13) are listed in Tables 1, 2 and 3.

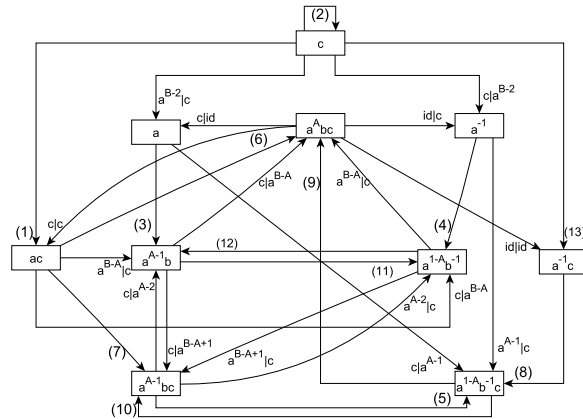
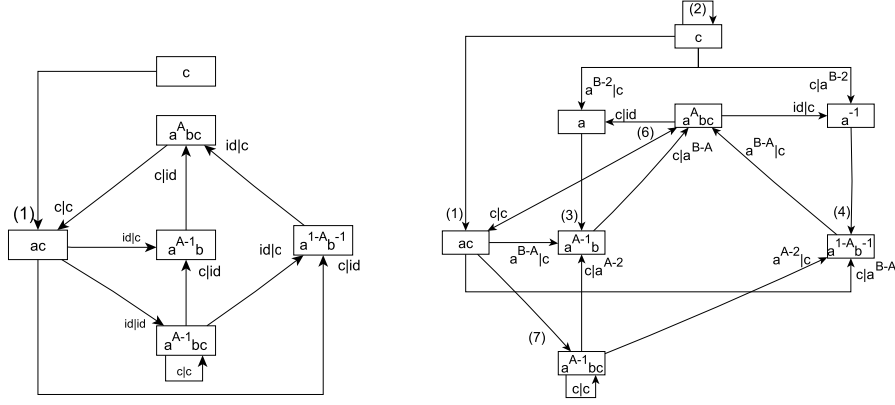


FIGURE 1. The graph $G(\mathcal{S}'')$ for $A \geq 3$ and $B \geq 5$ and $2A < B + 3$.


 (a) Proposition 4.1, case $A = 2, B = 2$

 (b) Proposition 4.1, case $A = 2, B \geq 3$

 FIGURE 2. The neighbor graph $G(\mathcal{S})$ of T

Edge	Labels		Name	Condition
$c \rightarrow ac$	a^{B-2} a^{B-3} \dots id	id a \dots a^{B-2}	(1)	$B \geq 2$ and $A \geq -1$
$c \rightarrow a^{-1}c$	a^{B-2} a^{B-3} \dots a^2	a^2 a^3 \dots a^{B-2}	(13)	$B \geq 4$ and $A \geq -1$
$c \rightarrow c$	a^{B-2} a^{B-3} \dots a	a a^2 \dots a^{B-2}	(2)	$B \geq 3$ and $A \geq -1$
$c \rightarrow a^{-1}$	c	a^{B-2}		$B \geq 2, A \geq -1$
$c \rightarrow a$	a^{B-2}	c		$B \geq 2, A \geq -1$
$a \rightarrow a^{A-1}b$	id a \dots a^{B-A-1}	a^{A-1} a^A \dots a^{B-2}	(3)	$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a \rightarrow a^{1-A}b^{-1}c$	c	a^{A-1}		$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a \rightarrow bc$	id	c		$B \geq 2, A \in \{-1, 0, 1\}$
$a \rightarrow a^{-1}bc$	a	c		$B \geq 3, A \in \{0, -1\}$
$a^{-1} \rightarrow a^{-1}bc$	c	a		$B \geq 3, A \in \{0, -1\}$
$a^{-1} \rightarrow bc$	c	id		$B \geq 3, A \in \{-1, 0, 1\}$

 Table 1. Edges of $G(\mathcal{S}'')$ (case $A \geq -1$ and $2A < B + 3$)

Edge	Labels		Name	Condition
$a^{-1} \rightarrow a^{1-A}b^{-1}$	a^{A-1} a^A \dots a^{B-2}	id a \dots a^{B-A+1}	(4)	$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$a^{-1} \rightarrow a^{1-A}b^{-1}c$	a^{A-1}	c		$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$abc \rightarrow a^{-1}bc$	id	id		$B \geq 2, A = 0$
$a^{A-1}bc \rightarrow a^{1-A}b^{-1}c$	a^{A-2} a^{A-3} \dots id	id a \dots a^{A-2}	(5)	$B \geq 2$ and $A \geq 2$
$a^{A-1}bc \rightarrow abc$	c	c		$B \geq 2, A \in \{-1, 0, 1\}$
$a^{A-1}bc \rightarrow a^{A-1}b$	c	a^{A-2}		$B \geq 2, A \geq 2$
$a^{A-1}bc \rightarrow a^{A-1}b$	c	a^{A-2}		$B \geq 2, A \geq 2$
$a^{A-1}bc \rightarrow a^{1-A}b^{-1}$	a^{A-2}	c		$B \geq 2, A \geq 2$
$ac \rightarrow a^A bc$	a^{B-A-1} a^{B-A-2} \dots id	id a \dots a^{B-A+1}	(6)	$B \geq 2, A \geq 1$ and $(A, B) \neq (2, 2)$
$ac \rightarrow a^{-1}bc$	a^{B-2} a^{B-3} \dots id	a^2 a^3 \dots a^{B-2}	(6)'	$B \geq 4, A \in \{0, -1\}$
$ac \rightarrow abc$	a^{B-2} a^{B-3} \dots id	id a \dots a^{B-2}	(14)	$B \geq 2, A = 0$
$ac \rightarrow a^{A-1}bc$	a^{B-A} a^{B-A-1} \dots id	id a \dots a^{B-A}	(7)	$B \geq 2$ and $A \geq 2$
$ac \rightarrow bc$	a^{B-2} a^{B-3} \dots a	a a^2 \dots a^{B-2}	(7)'	$B \geq 3, A \in \{-1, 0, 1\}$
$ac \rightarrow a^{A-1}b$	a^{B-A}	c		$B \geq 2, A \geq 2$
$ac \rightarrow a^{1-A}b^{-1}$	c	a^{B-A}		$B \geq 2, A \geq 2$
$a^A bc \rightarrow a^{-1}c$	id	id		$B \geq 2, A \geq -1$

Table 2. Edges of $G(\mathcal{S}'')$ (Case $A \geq -1$ and $2A < B + 3$)

Edge	Labels		Name	Condition
$a^A bc \rightarrow a$	c	id		$B \geq 2, A \geq -1$
$a^A bc \rightarrow a^{-1}$	id	c		$B \geq 2, A \geq -1$
$a^A bc \rightarrow ac$	c	c		$B \geq 2, A \geq -1$
$bc \rightarrow a^{-1}bc$	id	id		$B \geq 2, A = -1$
$a^{-1}c \rightarrow a^{1-A}b^{-1}c$	a^{B-2} a^{B-3} \dots a^A	a^A a^{A+1} \dots a^{B-2}	(8)	$B \geq A + 2, A > 0$
$a^{-1}c \rightarrow a^{-1}bc$	c	c		$B = 2, A = -1$
$a^{1-A}b^{-1}c \rightarrow a^A bc$	a^{B-2} a^{B-3} \dots a^{B-A+1}	a^{B-A+1} a^{B-A+2} \dots a^{B-2}	(9)	$B \geq 4$ and $A \geq 3$
$a^{1-A}b^{-1}c \rightarrow a^{A-1}bc$	a^{B-2} a^{B-3} \dots a^{B-A+2}	a^{B-A+2} a^{B-A+3} \dots a^{B-2}	(10)	$B \geq 6$ and $A \geq 4$
$a^{A-1}b \rightarrow a^{1-A}b^{-1}$	id a \dots a^{A-3}	a^{B-A+1} a^{B-A+2} \dots a^{B-2}	(11)	$B \geq 4$ and $A \geq 3$
$a^{A-1}b \rightarrow a^A bc$	c	a^{B-A}		$B \geq 2$ and $A \geq 2$
$a^{A-1}b \rightarrow a^{A-1}bc$	c	a^{B-A+1}		$B \geq 4$ and $A \geq 3$
$a^{1-A}b^{-1} \rightarrow a^{A-1}b$	a^{B-A+1} a^{B-A+2} \dots a^{B-2}	id a \dots a^{A-3}	(12)	$B \geq 4$ and $A \geq 3$
$a^{1-A}b^{-1} \rightarrow a^A bc$	a^{B-A}	c		$B \geq 2$ and $A \geq 2$
$a^{1-A}b^{-1} \rightarrow a^{A-1}bc$	a^{B-A+1}	c		$B \geq 4$ and $A \geq 3$

Table 3. Edges of $G(\mathcal{S}'')$ (Case $A \geq -1$ and $2A < B + 3$)

Since $\mathcal{S} \subset \mathcal{S}''$, it is clear that the neighbor graph $G(\mathcal{S})$ is a subgraph of the pseudo-neighbor graph. We will see that Characterization 3.5 will play an important role in the relationship between the neighbor graph $G(\mathcal{S})$ and the pseudo-neighbor graph $G(\mathcal{S}'')$.

Theorem 4.1. *Let \mathcal{S} be the neighbor set of T and \mathcal{S}'' be defined as in (4.1), (4.2) and (4.3). The following results hold for A, B satisfying $-1 \leq A \leq B$, $B \geq 2$ and $2A < B + 3$.*

(1) *For $A \geq 3$ and $B \geq 5$, $\mathcal{S} = \mathcal{S}''$, that is,*

$$\mathcal{S} = \{a, a^{-1}, a^{A-1}b, a^{1-A}b^{-1}, c, ac, a^{-1}c, a^{A-1}bc, a^{1-A}b^{-1}c, a^A bc\}.$$

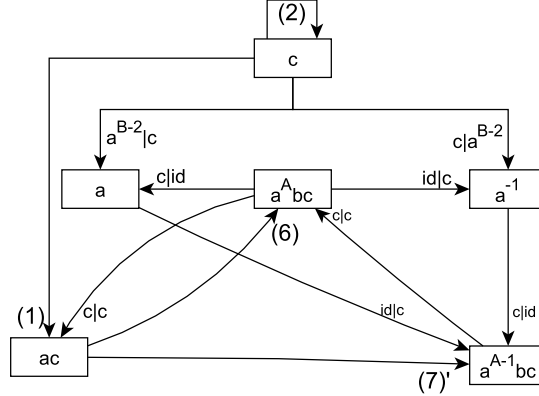


FIGURE 3. Proposition 4.1, Case $A = 1, B \geq 2$. We refer to Tables 1, 2 and 3 for the conditions on the edges.

- (2) For $A = 3$ and $B = 4$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c\}$, that is,
 $\mathcal{S} = \{a, a^{-1}, a^2b, a^{-2}b^{-1}, c, ac, a^2bc, a^{-2}b^{-1}c, a^3bc\}$.
- (3) For $A = 2$ and $B = 2$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c, a^{1-A}b^{-1}c, a, a^{-1}\}$, that is,
 $\mathcal{S} = \{ab, a^{-1}b^{-1}, c, ac, abc, a^2bc\}$.
- (4) For $A = 2$ and $B \geq 3$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c, a^{1-A}b^{-1}c\}$, that is,
 $\mathcal{S} = \{a, a^{-1}, ab, a^{-1}b^{-1}, c, ac, abc, a^2bc\}$.
- (5) For $A = 1$ and $B \geq 2$, $\mathcal{S} = \mathcal{S}'' \setminus \{a^{-1}c, a^{1-A}b^{-1}c, a^{A-1}b, a^{1-A}b^{-1}\}$, that is,
 $\mathcal{S} = \{a, a^{-1}, c, ac, abc, bc\}$.
- (6) For $A = 0$ and $B \geq 2$,
 $\mathcal{S} = \{a, a^{-1}, c, ac, a^{-1}bc, bc, abc\}$.
- (7) For $A = -1$ and $B = 2$,
 $\mathcal{S} = \{a, a^{-1}, c, a^{-1}c, a^{-1}bc, bc\};$
For $A = -1$ and $B \geq 3$,
 $\mathcal{S} = \{a, a^{-1}, c, ac, a^{-1}bc, bc\}.$

Proof. By Characterization 3.5, the neighbor graph $G(\mathcal{S})$ is obtained from the pseudo-neighbor graph $G(\mathcal{S}'')$ by deleting the states that are not the starting state of an infinite walk. For $A \geq 3, B \geq 5$, from Figure 1, it is clear that there is an infinite walk starting from each state of $G(\mathcal{S}'')$. For $A = 3, B = 4$, from Table 1, Table 2 and Table 3, we know that there is exactly one state $a^{-1}c$ from which there is no outgoing edge. For Item (3),(4),(5), and (6), see Figure 2(a), Figure 2(b), Figure 3 and Figure 5, respectively. For Item (7), it is easy to check that $a^{-1}c$ is the starting state of an infinite walk if and only if $B = 2$ and ac is the starting

state of an infinite walk if and only if $B \geq 3$. Since the neighbor set has at least six elements by Proposition 3.3, we get the results of Item (7) (see Figure 4 for more details). \square

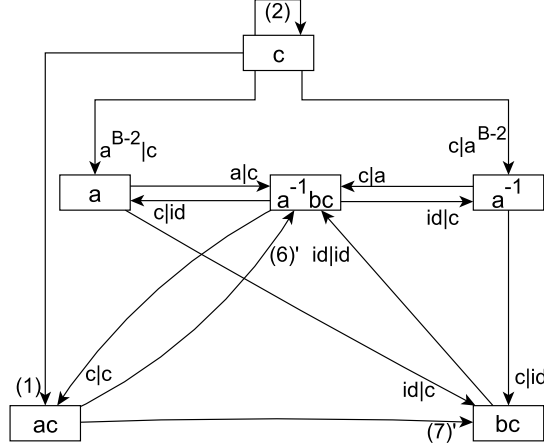


FIGURE 4. Proposition 4.1, case $A = -1, B \geq 3$. For the case $B = 2$, we only need to replace ac by $a^{-1}c$ and change the incoming and outgoing edges according to Tables 1, 2 and 3.

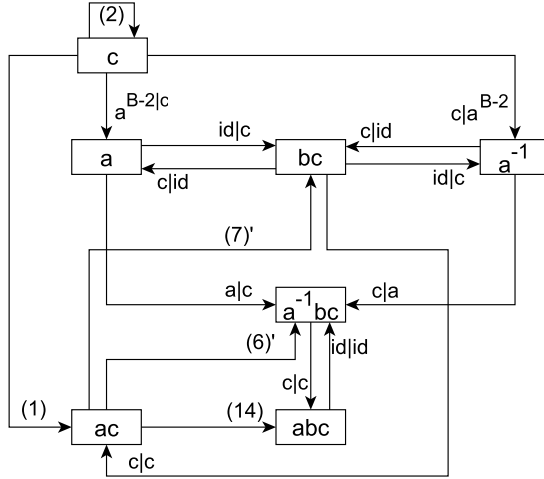


FIGURE 5. Proposition 4.1, the case $A = 0, B \geq 2$. We refer to Tables 1, 2 and 3 for the conditions on the edges.

5. CHARACTERIZATION OF THE DISK-LIKE TILES FOR $A \geq -1$ AND $2A < B + 3$

We are now in a position to study the topological properties of our family of $p2$ -tiles under the conditions $A \geq -1, 2A < B + 3$. We will characterize the disk-like tiles of the family under this condition. Loridant and Luo in [14] provided necessary and sufficient conditions for a $p2$ -tile to be disk-like. Before stating the theorem, we need a definition.

Definition 5.1. ([14]) If \mathcal{P} and \mathcal{F} are two sets of isometries in \mathbb{R}^2 , we say that \mathcal{P} is \mathcal{F} -connected iff for every disjoint pair (d, d') of elements in \mathcal{P} , there exist $n \geq 1$ and elements $d =: d_0, d_1, \dots, d_{n-1}, d_n := d'$ of \mathcal{P} such that $d_i^{-1}d_{i+1} \in \mathcal{F}$ for $i = 0, 1, \dots, n-1$.

The following statement is from [14]. In fact, the necessary part is due to the classification of Grünbaum and Shephard [7].

Proposition 5.2. *Let K be a crystal that tiles the plane by a $p2$ -group. Let \mathcal{F} be the corresponding digit set.*

- (1) *Suppose that the neighbor set \mathcal{S} of K has six elements. Then K is disk-like iff \mathcal{F} is \mathcal{S} -connected.*
- (2) *Suppose that the neighbor set \mathcal{S} of K has seven elements*

$$\{b^{\pm 1}, c, bc, a^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\},$$

where a, b are translations, and c is a π -rotation. Then K is disk-like iff \mathcal{F} is $\{b^{\pm 1}, c, bc, a^{-1}c\}$ -connected.

- (3) *Suppose that the neighbor set \mathcal{S} of K has eight elements*

$$\{b^{\pm 1}, c, bc, a^{-1}c, b^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\}$$

$$(\text{resp. } \{b^{\pm 1}, (a^{-1}b)^{\pm 1}, c, bc, ac, ab^{-1}c, a^{-1}bc\}),$$

where a, b are translations, and c is a π -rotation. Then K is disk-like iff \mathcal{F} is $\{b^{\pm 1}, c, a^{-1}c\}$ -(resp. $\{c, bc, ac, ab^{-1}c\}$)-connected.

- (4) *Suppose that the neighbor set \mathcal{S} of K has twelve elements*

$$\{a^{\pm 1}, b^{\pm 1}, (ab)^{\pm 1}, c, a^{-1}c, bc, abc, a^{-1}bc, a^{-1}b^{-1}c\},$$

where a, b are translations, and c is a π -rotation. Then K is disk-like iff \mathcal{F} is $\{c, a^{-1}c, bc\}$ -connected.

Applying this result, we obtain the following theorem.

Theorem 5.3. *Let $A, B \in \mathbb{Z}$ satisfy $-1 \leq A \leq B$, $B \geq 2$ and $2A < B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (2.2) and (2.3). Then the following statements hold.*

- (1) *If $A \in \{-1, 0, 1\}$, $B \geq 2$ or $A = 2$, $B = 2$, then T is disk-like.*
- (2) *If $A \geq 2$, $B \geq 3$, then T is non-disk-like.*

Proof. Let \mathcal{S} be the neighbor set of T . By Theorem 4.1, we know that in the assumption of $A \in \{-1, 1\}$, $B \geq 2$ and $A = 2$, $B = 2$, the neighbor sets of T all have six elements. Let us check the case $A = 1$, $B \geq 2$ by showing that \mathcal{D} is \mathcal{S} -connected and applying Proposition 5.2 (1). Then $A = -1$, $B \geq 2$ and $A = 2$, $B = 2$ can be checked in the same way.

For $A = 1$, $B \geq 2$, the digit set is $\mathcal{D} = \{id, a, \dots, a^{B-2}, c\}$ and the neighbor set is $\mathcal{S} = \{a, a^{-1}, c, abc, bc, ac\}$. It is easy to find that the disjoint pairs (d, d') in $\mathcal{D} \times \mathcal{D}$ are the following ones:

$$(5.1) \quad (id, a^\ell), (a^\ell, id), (id, c), (c, id), (a^k, a^{k'})(a^j, c), \text{ or } (c, a^j),$$

where $\ell, k, k', j \in \{1, 2, \dots, B-2\}$.

We will check the pair $(a^k, a^{k'})$ at first. If $k < k'$, then let $n = k' - k$, and

$$d_0 = a^k, d_1 = a^{k+1}, \dots, d_{n-1} = a^{k'-1}, d_n = a^{k'}.$$

hence $d_i^{-1}d_{i+1} = a$ is in \mathcal{S} for $0 \leq i \leq n-1$. If $k > k'$, $d_i^{-1}d_{i-1} = a^{-1}$ is also in \mathcal{S} for $0 \leq i \leq n-1$. To check (id, a^ℓ) and (a^j, c) , it suffices to check (id, a) and (a, c) . It is clear for (id, a) . For (a, c) , let $n = 2$, and $d_0 = a$, $d_1 = id$, $d_2 = c$. Hence, we have proved that \mathcal{D} is \mathcal{S} -connected. By Proposition 5.2 (1), T is disk-like.

For $A = 0$ and $B \geq 2$ and the neighbor set

$$\mathcal{S} = \{a, a^{-1}, c, a^{-1}bc, bc, ac, abc\}$$

has seven elements. By Proposition 5.2 (2), we need to prove that \mathcal{D} is $\{a, a^{-1}, c, ac, bc\}$ -connected. This is achieved in the same way as above.

We now prove Item (2). For $A = 2, B \geq 3$ and by Theorem 4.1, we know that

$$\mathcal{S} = \{a, a^{-1}, ab, a^{-1}b^{-1}, c, abc, a^2bc, ac\}.$$

Let $a' = a^2b, b' = ab$, then \mathcal{S} has the form

$$\Upsilon := \{b', b'^{-1}, a'^{-1}b', a'b'^{-1}, c, b'c, a'c, a'b'^{-1}c\}$$

of Proposition 5.2 (3). However, it is easily checked that \mathcal{D} is not $\{c, abc, ab^2c, ac\}$ -connected. By Proposition 5.2 (3), T is not disk-like.

For $A \geq 3, B \geq 4$, we have $\sharp\mathcal{S} = 9$ if $A = 3, B = 4$, and $\sharp\mathcal{S} = 10$ if $A \geq 3, B \geq 5$ by Theorem 4.1. According to Grünbaum and Shephard's classification of isohedral tilings (see [7, Sect. 6.2, p.285]), the cases in Proposition 5.2 are the only ones leading to disk-like $p2$ -tiles in the plane. So T is non-disk-like for $A \geq 3, B \geq 4$. \square

6. CHARACTERIZATION OF THE DISK-LIKE TILES FOR $A \leq -2$ AND $2|A| < B + 3$

We now deal with the case $A \leq -2$ and $2|A| < B + 3$. Let us recall a statement in [1, Equation (2.11), p. 2177]. Let T^ℓ be the lattice tile associated with the expanding matrix $M = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix}$ and digit set \mathcal{D} (see (2.3)) and \bar{T}^ℓ the lattice tile associated with the matrix $\bar{M} = \begin{pmatrix} 0 & -B \\ 1 & A \end{pmatrix}$ and \mathcal{D} . Then we have

$$(6.1) \quad \bar{T}^\ell = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T^\ell + \sum_{k=1}^{\infty} \bar{M}^{-2k} \begin{pmatrix} B-1 \\ 0 \end{pmatrix}.$$

It follows that T^ℓ and \bar{T}^ℓ have the same topology. It is remarkable that this does not hold for the associated crystiles T and \bar{T} , as is illustrated below.

By [3], we know all the information on the neighbor set of the lattice tile T^ℓ for $A \geq -1$, hence we can derive the neighbor set of \bar{T}^ℓ immediately.

Lemma 6.1. *If $2A < B + 3$ and $A > 0$, then the neighbor set of \bar{T}^ℓ is*

$$(6.2) \quad \{(-A, 1), (-A + 1, 1), (-1, 0), (1, 0), (A, -1), (A - 1, -1)\},$$

or, using translation mappings rather than vectors,

$$(6.3) \quad \{a^{-A}b, a^{-A+1}b, a^{-1}, a, a^Ab^{-1}, a^{A-1}b^{-1}\}.$$

Proof. The vector $\gamma = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2$ is a neighbor of T^ℓ iff $T^\ell \cap (T^\ell + \gamma) \neq \emptyset$. Let $\gamma' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$, then this is equivalent to $\bar{T}^\ell \cap (\bar{T}^\ell + \gamma') \neq \emptyset$ by (6.1). Thus, using Proposition 3.1, we get the neighbor set (6.2) of \bar{T}^ℓ . \square

For $-1 \leq A \leq B \geq 2$, the data $(g, \mathcal{D}, p2)$ is a crystallographic number system, hence, the tiling group is the whole crystallographic group $p2$ [13]. It follows from Lemma 2.5 that this property still holds for $A \leq -2$. Now, by Lemma 6.1, to obtain the neighbor set of $p2$ -crystiles for $A \leq -2$, we only need to repeat the methods in Section 3 and 4, dealing with similar estimates and computations. We come to the following theorem for $A \leq -2$ (we do not reproduce the computations).

Theorem 6.2. *Let $A, B \in \mathbb{Z}$ satisfy $2 \leq -A \leq B$ and $2|A| < B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (2.2) and (2.3). Then the following statements hold.*

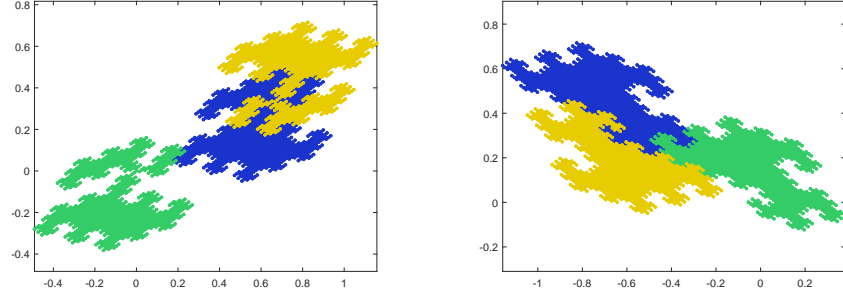


FIGURE 6. T for $A = 2, B = 3$ on the left and for $A = -2, B = 3$ on the right

- (1) For $A = -2$ and $B = 2$ or 3 , the neighbor set of the crystile T is

$$\mathcal{S} = \{a, a^{-1}, c, a^{-1}c, a^{-2}bc, a^{-1}bc\};$$

- (2) For $A = -2, B \geq 4$, the neighbor set of the crystile T is

$$\mathcal{S} = \{a, a^{-1}, c, ac, a^{-2}bc, a^{-1}bc\}.$$

- (3) For $A = -3, B = 4$, the neighbor set of the crystile T is

$$\mathcal{S} = \{a, a^{-1}, a^{-2}b, a^2b^{-1}, c, a^{-1}c, a^{-2}bc, a^{-3}bc\}.$$

- (4) For $A = -3, B \geq 5$, the neighbor set of the crystile T is

$$\mathcal{S} = \{a, a^{-1}, a^{A+1}b, a^{-1-A}b^{-1}, c, ac, a^{A+1}bc, a^A bc, a^{-1}c\}.$$

- (5) For $A = -4, B \geq 6$, the neighbor set of the crystile T is

$$\mathcal{S} = \{a, a^{-1}, a^{A+1}b, a^{-1-A}b^{-1}, c, a^{-1}c, ac, a^{A+1}bc, a^A bc, a^{-A-1}b^{-1}c\}.$$

Consequently, we can infer from Lemma 5.2 the following theorem.

Theorem 6.3. *Let $A, B \in \mathbb{Z}$ satisfy $2 \leq -A \leq B$ and $2|A| < B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (2.2) and (2.3). Then the following statements hold.*

- (1) *If $A = -2, B \geq 2$, then T is disk-like.*
- (2) *If $A \leq -3, B \geq 4$, then T is not disk-like.*

Proof. For Item (1), we know from Theorem 6.2 that the neighbor set of T has six neighbors. Thus, by Proposition 5.2 Item (1), T is disk-like.

For $A = -3, B = 4$, the neighbor set is

$$\mathcal{S} = \{a, a^{-1}, a^{-2}b, a^2b^{-1}, c, a^{-2}bc, a^{-3}bc, a^{-1}c\}.$$

Let $a' = a^{-3}b, b' = a^{-1}$, then \mathcal{S} has the form

$$\Upsilon := \{b', b'^{-1}, a'^{-1}b', a'b'^{-1}, c, b'c, a'c, a'b'^{-1}c\}$$

of Proposition 5.2 (3). However, it is easily checked that \mathcal{D} is not $\{c, a^{-2}bc, ab^{-3}c, a^{-1}c\}$ -connected. By Proposition 5.2 Item (3), T is not disk-like.

For the cases $A = -3, B \geq 5$ and $A \leq -4, B \geq 6$, T has 9 and 10 neighbours, respectively. Thus T is not disk-like as we have discussed in Theorem 5.3. \square

In particular, we see that for $A = 2$ and $B = 3$, the crystile is not disk-like (from Theorem 5.3), while for $A = -2$ and $B = 3$, it is disk-like (see Figure 6).

7. NON-DISK-LIKENESS OF TILES FOR $2|A| \geq B + 3$

So far, we have dealt with the case $2|A| < B + 3$ and characterized the disk-like $p2$ -tiles in Theorem 5.3 and Theorem 6.3. If $2|A| \geq B + 3$, it was proved in [12] that the lattice tiles T^ℓ are not disk-like. We prove that this also holds for the corresponding $p2$ -tiles T .

Recall that the $p2$ -tile T satisfies the equation

$$(7.1) \quad T = \bigcup_{i=1}^B f_i(T),$$

where

$$f_1 = g^{-1} \circ id, \quad f_i = g^{-1} \circ a^{i-1} \quad (2 \leq i \leq B-1), \quad f_B = g^{-1} \circ c,$$

g is the expanding map, and \mathcal{D} is the digit set defined as before. We denote the fixed point of a mapping f by $\text{Fix}(f)$ and the linear part of g by M . Then we have the following facts:

$$(7.2) \quad \text{Fix}(f_i) = (M - I_2)^{-1} \begin{pmatrix} i-1 - \frac{B-1}{2} \\ 0 \end{pmatrix} \quad \text{for } 1 \leq i \leq B-1,$$

$$(7.3) \quad \text{Fix}(f_B) = (M + I_2)^{-1} \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}.$$

By (7.1), the fixed points given by (7.2) and (7.3) all belong to T . First of all, we give a key lemma for the main result.

Lemma 7.1. *Let $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $2|A| \geq B + 3$, and let T be the $p2$ -crystile defined by (2.2) and (2.3) and $c(T)$ be the π -rotation of T . Then $\#(T \cap c(T)) \geq 2$.*

Proof. By (7.2), we notice that for $2 \leq p, q \leq B-2$

$$\text{Fix}(f_p) = -\text{Fix}(f_q) \text{ if } p + q = B - 1.$$

This means that $\text{Fix}(f_p)$ and $\text{Fix}(f_q)$ are both in T and $c(T)$. If $B > 3$, these points are different and we are done. If $B \leq 3$, we only need to consider the case $|A| = 3, B = 3$ since we assume that $2|A| \geq B + 3$. Since $B = 3$, by (7.2), $\text{Fix}(f_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which is in $T \cap c(T)$. And for the case $A = 3, B = 3$, there exists an eventually periodic sequence of edges (see Figure 7).

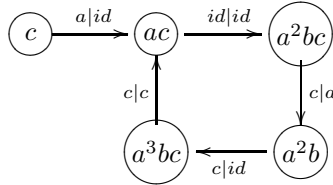


FIGURE 7. An eventually periodic sequence of edges for $A = 3, B = 3$.

The edges of this figure are defined in the same way as in Definition 3.4 and it follows that

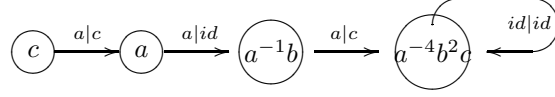
$$x_0 = \lim_{n \rightarrow \infty} g^{-1}a \circ (g^{-1} \circ g^{-1}c \circ g^{-1}c \circ g^{-1}c)^n(t) \in T \cap c(T),$$

(see also Characterization 3.5). Here, $t \in \mathbb{R}^2$ is arbitrary. Note that

$$x_0 = g^{-1}a \left(\text{Fix}(g^{-1} \circ g^{-1}c \circ g^{-1}c \circ g^{-1}c) \right),$$

and it is easy to compute that $x_0 = \begin{pmatrix} -\frac{13}{73} \\ \frac{16}{219} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For the case $A = -3, B = 3$, we find the eventually periodic sequence of edges



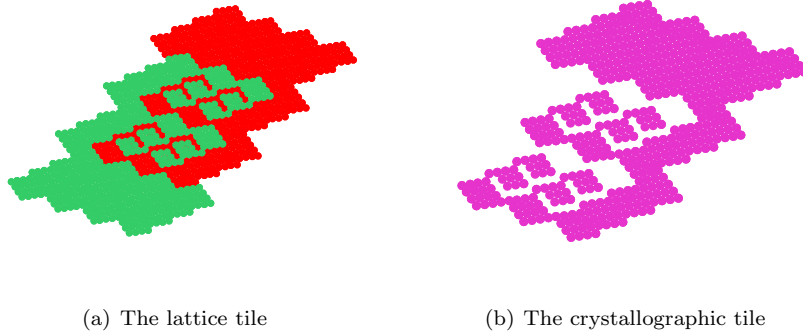
So we have

$$x'_0 = \lim_{n \rightarrow \infty} g^{-1}a \circ g^{-1}a \circ g^{-1}a \circ (g^{-1})^n(t) \in T \cap c(T),$$

and it is easy to verify that $x'_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. \square

Theorem 7.2. *Let $A, B \in \mathbb{Z}$ satisfying $|A| \leq B$ and $2|A| \geq B + 3$, and let T be the crystallographic replication tile defined by the data (g, \mathcal{D}) given in (2.2) and (2.3). Then T is not disk-like.*

Proof. By a result of [12], we know that if $2|A| \geq B + 3$, then T^ℓ is not disk-like. Suppose that T is disk-like. By Lemma 7.1, we have $\sharp(T \cap c(T)) \geq 2$. By [15, Proposition 4.1 item (2), p. 127], this implies that $T \cap c(T)$ is a simple arc. Therefore $T \cup c(T)$ is disk-like, as the union of two topological disks whose intersection is a simple arc is again a topological disk. However, by Lemma 2.4, T^ℓ is a translation of $T \cup c(T)$, therefore T^ℓ must be disk-like. This contradicts the assumption $2|A| \geq B + 3$. \square



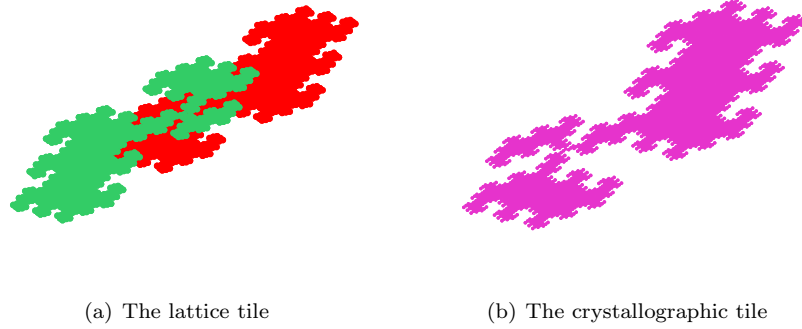
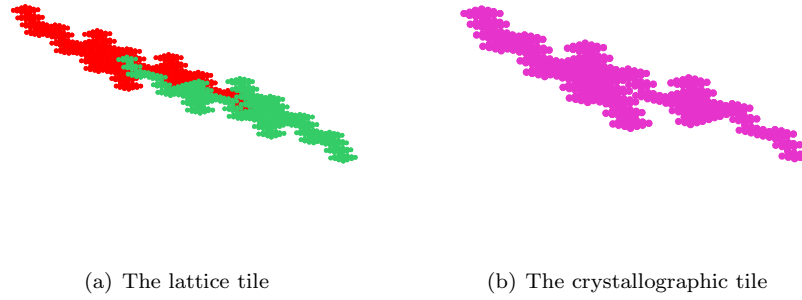
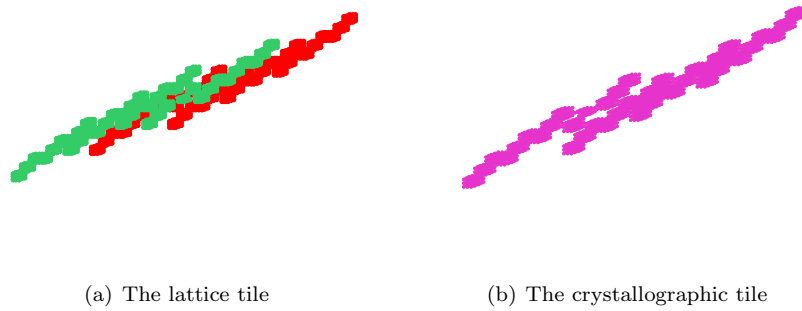
(a) The lattice tile

(b) The crystallographic tile

FIGURE 8. $A = 1, B = 4$.

8. EXAMPLES

Now we provide some examples. For fixed A and B , even though the lattice tile T^ℓ is a translate of $T \cup (-T)$, T and T^ℓ may have completely different topological behaviour. We give the following examples to illustrate this phenomenon. In Figure 8, $A = 1, B = 4$, T and T^ℓ are both disk-like. For Figure 9 and Figure 10, T^ℓ is disk-like while T is not. In Figure 11, T and T^ℓ are both not disk-like.


 FIGURE 9. Lattice tile and Crystile for $A = 2, B = 3$.

 FIGURE 10. Lattice tile and Crystile for $A = -3, B = 4$.

 FIGURE 11. Lattice tile and Crystile for $A = 3, B = 3$.

REFERENCES

- [1] S. AKIYAMA AND B. LORIDANT, *Boundary parametrization of planar self-affine tiles with collinear digit set*, Sci. China Math., 53 (2010), pp. 2173–2194.
- [2] ———, *Boundary parametrization of self-affine tiles*, J. Math. Soc. Japan, 63 (2011), pp. 525–579.
- [3] S. AKIYAMA AND J. M. THUSWALDNER, *The topological structure of fractal tilings generated by quadratic number systems*, Comput. Math. Appl., 49 (2005), pp. 1439–1485.

- [4] C. BANDT AND Y. WANG, *Disk-like self-affine tiles in \mathbb{R}^2* , Discrete Comput. Geom., 26 (2001), pp. 591–601.
- [5] J. J. BURCKHARDT, *Die Bewegungsgruppen der Kristallographie*, vol. 13 of Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Verlag Birkhäuser, Basel, 1947.
- [6] G. GELBRICH, *Crystallographic reptiles*, Geom. Dedicata, 51 (1994), pp. 235–256.
- [7] B. GRÜNBAUM AND G. C. SHEPHARD, *Tilings and patterns*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Company, New York, 1989. An introduction.
- [8] I. KÁTAI, *Number systems and fractal geometry*, University of Pécs, (1995).
- [9] I. KIRAT AND K.-S. LAU, *On the connectedness of self-affine tiles*, J. London Math. Soc. (2), 62 (2000), pp. 291–304.
- [10] J. C. LAGARIAS AND Y. WANG, *Integral self-affine tiles in \mathbb{R}^n . I. Standard and nonstandard digit sets*, J. London Math. Soc. (2), 54 (1996), pp. 161–179.
- [11] ———, *Self-affine tiles in \mathbb{R}^n* , Adv. Math., 121 (1996), pp. 21–49.
- [12] K.-S. LEUNG AND K.-S. LAU, *Disklikeness of planar self-affine tiles*, Trans. Amer. Math. Soc., 359 (2007), pp. 3337–3355.
- [13] B. LORIDANT, *Crystallographic number systems*, Monatsh Math, 167 (2012), pp. 511–529.
- [14] B. LORIDANT AND J. LUO, *On a theorem of Bandt and Wang and its extension to p_2 -tiles*, Discrete Comput. Geom., 41 (2009), pp. 616–642.
- [15] B. LORIDANT, J. LUO, AND J. M. THUSWALDNER, *Topology of crystallographic tiles*, Geom. Dedicata, 128 (2007), pp. 113–144.
- [16] K. SCHEICHER AND J. M. THUSWALDNER, *Canonical number systems, counting automata and fractals*, Math. Proc. Cambridge Philos. Soc., 133 (2002), pp. 163–182.

CHAIR OF MATHEMATICS AND STATISTICS, UNIVERSITY OF LEOBEN, FRANZ-JOSEF-STRASSE 18,
A-8700 LEOBEN, AUSTRIA

E-mail address: benoit.loridant@unileoben.ac.at

E-mail address: zhangsq-ccnu@sina.com